Chapter 3

Vectors

CHAPTER OUTLINE

▲ *These controls in the cockpit of a commercial aircraft assist the pilot in maintaining control over the* velocity *of the aircraft—how fast it is traveling and in what direction it is traveling—allowing it to land safely. Quantities that are defined by both a magnitude and a direction, such as velocity, are called* vector quantities. *(Mark Wagner/Getty Images)*

In our study of physics, we often need to work with physical quantities that have both numerical and directional properties. As noted in Section 2.1, quantities of this nature are vector quantities. This chapter is primarily concerned with vector algebra and with some general properties of vector quantities. We discuss the addition and subtraction of vector quantities, together with some common applications to physical situations.

Vector quantities are used throughout this text, and it is therefore imperative that you master both their graphical and their algebraic properties.

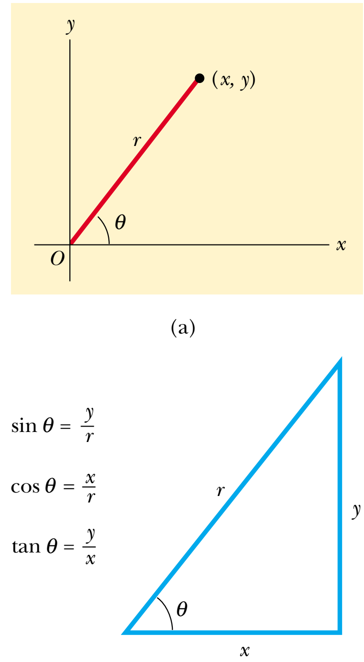
3.1 Coordinate Systems

Many aspects of physics involve a description of a location in space. In Chapter 2, for example, we saw that the mathematical description of an object's motion requires a method for describing the object's position at various times. This description is accomplished with the use of coordinates, and in Chapter 2 we used the Cartesian coordinate system, in which horizontal and vertical axes intersect at a point defined as the origin (Fig. 3.1). Cartesian coordinates are also called *rectangular coordinates.*

Sometimes it is more convenient to represent a point in a plane by its *plane polar coordinates* (r, θ), as shown in Figure 3.2a. In this *polar coordinate system,* r is the distance from the origin to the point having Cartesian coordinates (x, y), and θ is the angle between a line drawn from the origin to the point and a fixed axis. This fixed axis is usually the positive x axis, and θ is usually measured counterclockwise from it. From the right triangle in Figure 3.2b, we find that $\sin\theta = y/r$ and that $\cos\theta = x/r$. (A review of trigonometric functions is given in Appendix B.4.) Therefore, starting with the plane polar coordinates of any point, we can obtain the Cartesian coordinates by using the equations

$$x = r\cos\theta \tag{3.1}$$

$$y = r\sin\theta \tag{3.2}$$



Active Figure 3.2 (a) The plane polar coordinates of a point are represented by the distance r and the angle θ, where θ is measured counterclockwise from the positive x axis. (b) The right triangle used to relate (x, y) to (r, θ).

At the Active Figures link at http://www.pse6.com, you can move the point and see the changes to the rectangular and polar coordinates as well as to the sine, cosine, and tangent of angle θ.

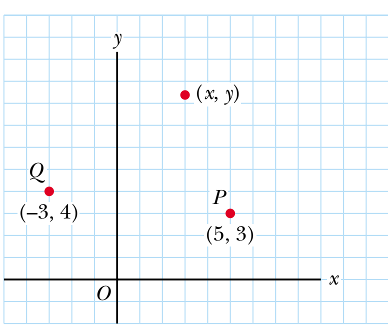


Figure 3.1 Designation of points in a Cartesian coordinate system. Every point is labeled with coordinates (x, y).

Furthermore, the definitions of trigonometry tell us that

$$\tan\theta = \frac{y}{x} \tag{3.3}$$

$$r = \sqrt{x^2 + y^2} \tag{3.4}$$

Equation 3.4 is the familiar Pythagorean theorem.

These four expressions relating the coordinates (x, y) to the coordinates (r, θ) apply only when θ is defined as shown in Figure 3.2a—in other words, when positive θ is an angle measured counterclockwise from the positive x axis. (Some scientific calculators perform conversions between Cartesian and polar coordinates based on these standard conventions.) If the reference axis for the polar angle θ is chosen to be one other than the positive x axis or if the sense of increasing θ is chosen differently, then the expressions relating the two sets of coordinates will change.

Example 3.1 Polar Coordinates

The Cartesian coordinates of a point in the xy plane are $(x, y) = (-3.50, -2.50)$ m, as shown in Figure 3.3. Find the polar coordinates of this point.

Solution For the examples in this and the next two chapters we will illustrate the use of the General Problem-Solving Strategy outlined at the end of Chapter 2. In subsequent chapters, we will make fewer explicit references to this strategy, as you will have become familiar with it and should be applying it on your own. The drawing in Figure 3.3 helps us to *conceptualize* the problem. We can *categorize* this as a plug-in problem. From Equation 3.4,

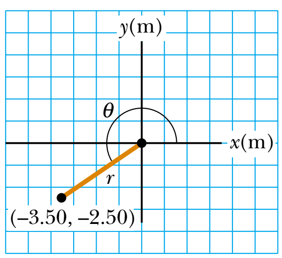
$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} = \boxed{4.30\text{ m}}$$

and from Equation 3.3,

$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = \boxed{216°}$$

Note that you must use the signs of x and y to find that the point lies in the third quadrant of the coordinate system. That is, $\theta = 216°$ and not $35.5°$.



Active Figure 3.3 (Example 3.1) Finding polar coordinates when Cartesian coordinates are given.

At the Active Figures link at http://www.pse6.com, you can move the point in the xy plane and see how its Cartesian and polar coordinates change.

3.2 Vector and Scalar Quantities

As noted in Chapter 2, some physical quantities are scalar quantities whereas others are vector quantities. When you want to know the temperature outside so that you will know how to dress, the only information you need is a number and the unit "degrees C" or "degrees F." Temperature is therefore an example of a *scalar quantity*:

A **scalar quantity** is completely specified by a single value with an appropriate unit and has no direction.

Other examples of scalar quantities are volume, mass, speed, and time intervals. The rules of ordinary arithmetic are used to manipulate scalar quantities.

If you are preparing to pilot a small plane and need to know the wind velocity, you must know both the speed of the wind and its direction. Because direction is important for its complete specification, velocity is a *vector quantity*:

A **vector quantity** is completely specified by a number and appropriate units plus a direction.

Another example of a vector quantity is displacement, as you know from Chapter 2. Suppose a particle moves from some point Ⓐ to some point Ⓑ along a straight path, as shown in Figure 3.4. We represent this displacement by drawing an arrow from Ⓐ to Ⓑ, with the tip of the arrow pointing away from the starting point. The direction of the arrowhead represents the direction of the displacement, and the length of the arrow represents the magnitude of the displacement. If the particle travels along some other path from Ⓐ to Ⓑ, such as the broken line in Figure 3.4, its displacement is still the arrow drawn from Ⓐ to Ⓑ. Displacement depends only on the initial and final positions, so the displacement vector is independent of the path taken between these two points.

In this text, we use a boldface letter, such as **A**, to represent a vector quantity. Another notation is useful when boldface notation is difficult, such as when writing on paper or on a chalkboard—an arrow is written over the symbol for the vector: $\vec{\mathbf{A}}$. The magnitude of the vector **A** is written either A or $|\mathbf{A}|$. The magnitude of a vector has physical units, such as meters for displacement or meters per second for velocity. The magnitude of a vector is *always* a positive number.

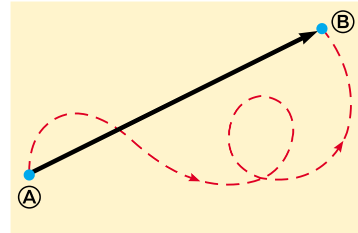


Figure 3.4 As a particle moves from Ⓐ to Ⓑ along an arbitrary path represented by the broken line, its displacement is a vector quantity shown by the arrow drawn from Ⓐ to Ⓑ.

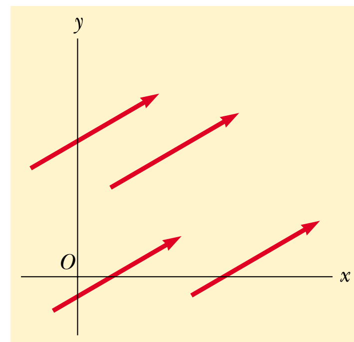


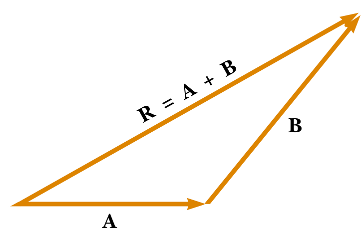
Figure 3.5 These four vectors are equal because they have equal lengths and point in the same direction.

Quick Quiz 3.1 Which of the following are vector quantities and which are scalar quantities? (a) your age (b) acceleration (c) velocity (d) speed (e) mass

3.3 Some Properties of Vectors

Equality of Two Vectors

For many purposes, two vectors **A** and **B** may be defined to be equal if they have the same magnitude and point in the same direction. That is, $\mathbf{A} = \mathbf{B}$ only if $A = B$ and if **A** and **B** point in the same direction along parallel lines. For example, all the vectors in Figure 3.5 are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.



Active Figure 3.6 When vector **B** is added to vector **A**, the resultant **R** is the vector that runs from the tail of **A** to the tip of **B**.

Go to the Active Figures link at http://www.pse6.com.

Adding Vectors

The rules for adding vectors are conveniently described by graphical methods. To add vector **B** to vector **A**, first draw vector **A** on graph paper, with its magnitude represented by a convenient length scale, and then draw vector **B** to the same scale with its tail starting from the tip of **A**, as shown in Figure 3.6. The **resultant vector** $\mathbf{R} = \mathbf{A} + \mathbf{B}$ is the vector drawn from the tail of **A** to the tip of **B**.

For example, if you walked 3.0 m toward the east and then 4.0 m toward the north, as shown in Figure 3.7, you would find yourself 5.0 m from where you started, measured at an angle of 53° north of east. Your total displacement is the vector sum of the individual displacements.

A geometric construction can also be used to add more than two vectors. This is shown in Figure 3.8 for the case of four vectors. The resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$ is the vector that completes the polygon. In other words, **R is the vector drawn from the tail of the first vector to the tip of the last vector**.

When two vectors are added, the sum is independent of the order of the addition. (This fact may seem trivial, but as you will see in Chapter 11, the order is important

PITFALL PREVENTION

3.1 Vector Addition versus Scalar Addition

Keep in mind that $\mathbf{A} + \mathbf{B} = \mathbf{C}$ is very different from $A + B = C$. The first is a vector sum, which must be handled carefully, such as with the graphical method described here. The second is a simple algebraic addition of numbers that is handled with the normal rules of arithmetic.

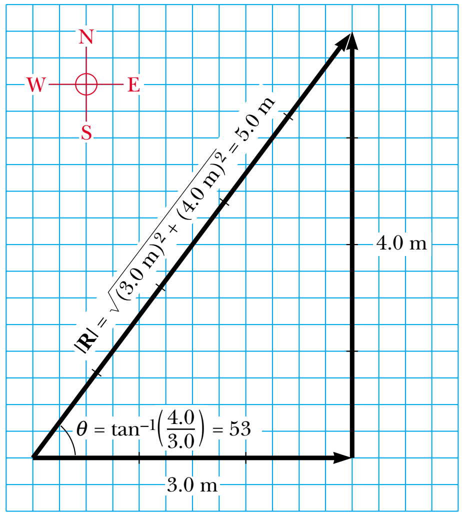


Figure 3.7 Vector addition. Walking first 3.0 m due east and then 4.0 m due north leaves you 5.0 m from your starting point.

when vectors are multiplied). This can be seen from the geometric construction in Figure 3.9 and is known as the **commutative law of addition:**

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{3.5}$$

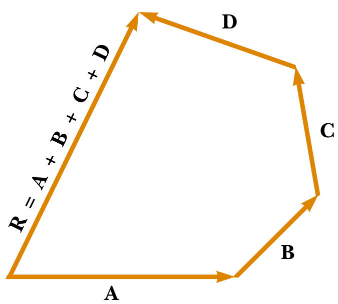


Figure 3.8 Geometric construction for summing four vectors. The resultant vector **R** is by definition the one that completes the polygon.

When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together. A geometric proof of this rule for three vectors is given in Figure 3.10. This is called the **associative law of addition:**

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{3.6}$$

In summary, **a vector quantity has both magnitude and direction and also obeys the laws of vector addition** as described in Figures 3.6 to 3.10. When two or more vectors are added together, all of them must have the same units and all of them must be the same type of quantity. It would be meaningless to add a velocity vector (for example, 60 km/h to the east) to a displacement vector (for example, 200 km to the north) because they represent different physical quantities. The same rule also applies to scalars. For example, it would be meaningless to add time intervals to temperatures.

Negative of a Vector

The negative of the vector **A** is defined as the vector that when added to **A** gives zero for the vector sum. That is, $\mathbf{A} + (-\mathbf{A}) = 0$. The vectors **A** and $-\mathbf{A}$ have the same magnitude but point in opposite directions.

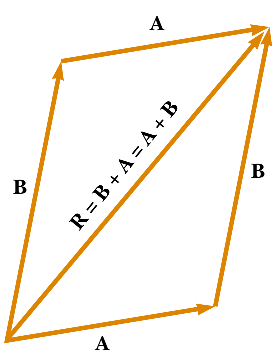


Figure 3.9 This construction shows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$—in other words, that vector addition is commutative.

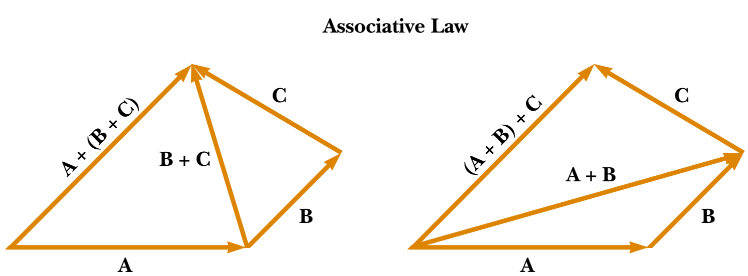


Figure 3.10 Geometric constructions for verifying the associative law of addition.

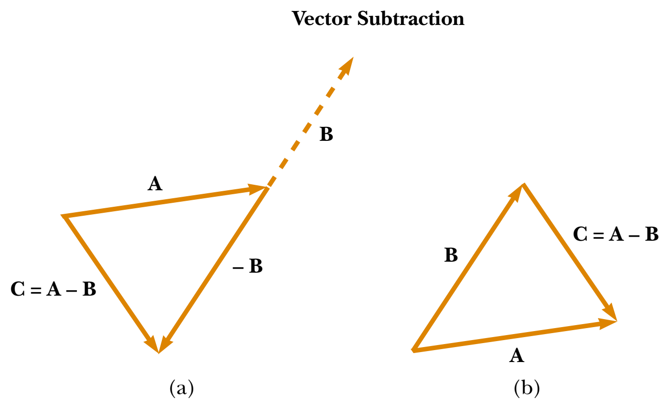


Figure 3.11 (a) This construction shows how to subtract vector **B** from vector **A**. The vector $-\mathbf{B}$ is equal in magnitude to vector **B** and points in the opposite direction. To subtract **B** from **A**, apply the rule of vector addition to the combination of **A** and $-\mathbf{B}$: Draw **A** along some convenient axis, place the tail of $-\mathbf{B}$ at the tip of **A**, and **C** is the difference $\mathbf{A} - \mathbf{B}$. (b) A second way of looking at vector subtraction. The difference vector $\mathbf{C} = \mathbf{A} - \mathbf{B}$ is the vector that we must add to **B** to obtain **A**.

Subtracting Vectors

The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation $\mathbf{A} - \mathbf{B}$ as vector $-\mathbf{B}$ added to vector **A**:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{3.7}$$

The geometric construction for subtracting two vectors in this way is illustrated in Figure 3.11a.

Another way of looking at vector subtraction is to note that the difference $\mathbf{A} - \mathbf{B}$ between two vectors **A** and **B** is what you have to add to the second vector to obtain the first. In this case, the vector $\mathbf{A} - \mathbf{B}$ points from the tip of the second vector to the tip of the first, as Figure 3.11b shows.

Quick Quiz 3.2 The magnitudes of two vectors **A** and **B** are $A = 12$ units and $B = 8$ units. Which of the following pairs of numbers represents the *largest* and *smallest* possible values for the magnitude of the resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$? (a) 14.4 units, 4 units (b) 12 units, 8 units (c) 20 units, 4 units (d) none of these answers.

Quick Quiz 3.3 If vector **B** is added to vector **A**, under what condition does the resultant vector $\mathbf{A} + \mathbf{B}$ have magnitude $A + B$? (a) **A** and **B** are parallel and in the same direction. (b) **A** and **B** are parallel and in opposite directions. (c) **A** and **B** are perpendicular.

Quick Quiz 3.4 If vector **B** is added to vector **A**, which *two* of the following choices must be true in order for the resultant vector to be equal to zero? (a) **A** and **B** are parallel and in the same direction. (b) **A** and **B** are parallel and in opposite directions. (c) **A** and **B** have the same magnitude. (d) **A** and **B** are perpendicular.

Example 3.2 A Vacation Trip

A car travels 20.0 km due north and then 35.0 km in a direction 60.0° west of north, as shown in Figure 3.12a. Find the magnitude and direction of the car's resultant displacement.

Solution The vectors **A** and **B** drawn in Figure 3.12a help us to *conceptualize* the problem. We can *categorize* this as a relatively simple analysis problem in vector addition. The displacement **R** is the resultant when the two individual displacements **A** and **B** are added. We can further categorize this as a problem about the analysis of triangles, so we appeal to our expertise in geometry and trigonometry.

In this example, we show two ways to *analyze* the problem of finding the resultant of two vectors. The first way is to solve the problem geometrically, using graph paper and a protractor to measure the magnitude of **R** and its direction in Figure 3.12a. (In fact, even when you know you are going to be carrying out a calculation, you should sketch the vectors to check your results.) With an ordinary ruler and protractor, a large diagram typically gives answers to two-digit but not to three-digit precision.

The second way to solve the problem is to analyze it algebraically. The magnitude of **R** can be obtained from the law of cosines as applied to the triangle (see Appendix B.4). With $\theta = 180° - 60° = 120°$ and $R^2 = A^2 + B^2 - 2AB\cos\theta$, we find that

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$$= \sqrt{(20.0\text{ km})^2 + (35.0\text{ km})^2 - 2(20.0\text{ km})(35.0\text{ km})\cos 120°}$$

$$= \boxed{48.2\text{ km}}$$

The direction of **R** measured from the northerly direction can be obtained from the law of sines (Appendix B.4):

$$\frac{\sin\beta}{B} = \frac{\sin\theta}{R}$$

$$\sin\beta = \frac{B}{R}\sin\theta = \frac{35.0\text{ km}}{48.2\text{ km}}\sin 120° = 0.629$$

$$\boxed{\beta = 39.0°}$$

The resultant displacement of the car is 48.2 km in a direction 39.0° west of north.

We now *finalize* the problem. Does the angle β that we calculated agree with an estimate made by looking at Figure 3.12a or with an actual angle measured from the diagram using the graphical method? Is it reasonable that the magnitude of **R** is larger than that of both **A** and **B**? Are the units of **R** correct?

While the graphical method of adding vectors works well, it suffers from two disadvantages. First, some individuals find using the laws of cosines and sines to be awkward. Second, a triangle only results if you are adding two vectors. If you are adding three or more vectors, the resulting geometric shape is not a triangle. In Section 3.4, we explore a new method of adding vectors that will address both of these disadvantages.

What If? **Suppose the trip were taken with the two vectors in reverse order: 35.0 km at 60.0° west of north first, and then 20.0 km due north. How would the magnitude and the direction of the resultant vector change?**

Answer They would not change. The commutative law for vector addition tells us that the order of vectors in an addition is irrelevant. Graphically, Figure 3.12b shows that the vectors added in the reverse order give us the same resultant vector.

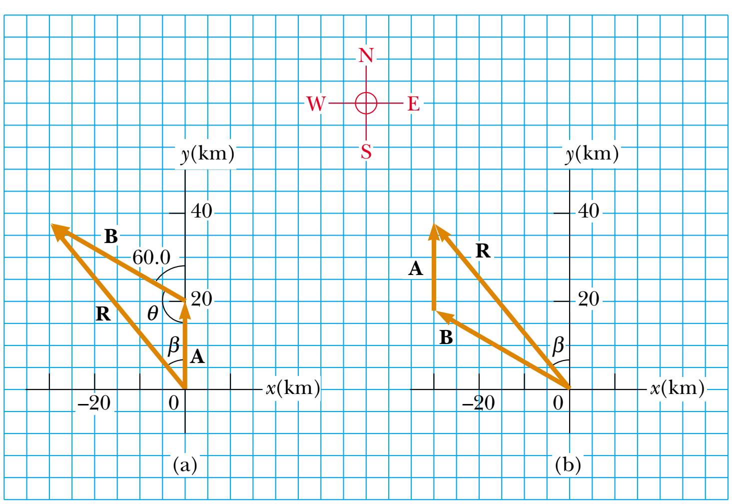


Figure 3.12 (Example 3.2) (a) Graphical method for finding the resultant displacement vector **R** = **A** + **B**. (b) Adding the vectors in reverse order (**B** + **A**) gives the same result for **R**.

Multiplying a Vector by a Scalar

If vector **A** is multiplied by a positive scalar quantity m, then the product $m\mathbf{A}$ is a vector that has the same direction as **A** and magnitude mA. If vector **A** is multiplied by a negative scalar quantity $-m$, then the product $-m\mathbf{A}$ is directed opposite **A**. For example, the vector $5\mathbf{A}$ is five times as long as **A** and points in the same direction as **A**; the vector $-\frac{1}{3}\mathbf{A}$ is one-third the length of **A** and points in the direction opposite **A**.

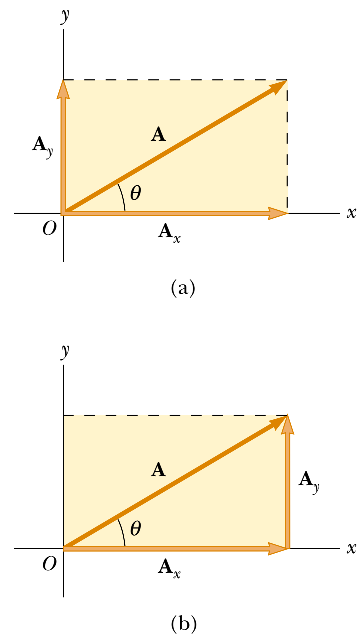


Figure 3.13 (a) A vector **A** lying in the xy plane can be represented by its component vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. (b) The y component vector $\mathbf{A}_y$ can be moved to the right so that it adds to $\mathbf{A}_x$. The vector sum of the component vectors is **A**. These three vectors form a right triangle.

3.4 Components of a Vector and Unit Vectors

The graphical method of adding vectors is not recommended whenever high accuracy is required or in three-dimensional problems. In this section, we describe a method of adding vectors that makes use of the projections of vectors along coordinate axes. These projections are called the **components** of the vector. Any vector can be completely described by its components.

Consider a vector **A** lying in the xy plane and making an arbitrary angle θ with the positive x axis, as shown in Figure 3.13a. This vector can be expressed as the sum of two other vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. From Figure 3.13b, we see that the three vectors form a right triangle and that $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$. We shall often refer to the "components of a vector **A**," written A_x and A_y (without the boldface notation). The component A_x represents the projection of **A** along the x axis, and the component A_y represents the projection of **A** along the y axis. These components can be positive or negative. The component A_x is positive if $\mathbf{A}_x$ points in the positive x direction and is negative if $\mathbf{A}_x$ points in the negative x direction. The same is true for the component A_y.

From Figure 3.13 and the definition of sine and cosine, we see that $\cos\theta = A_x/A$ and that $\sin\theta = A_y/A$. Hence, the components of **A** are

Components of the vector A

$$A_x = A\cos\theta \tag{3.8}$$

$$A_y = A\sin\theta \tag{3.9}$$

These components form two sides of a right triangle with a hypotenuse of length A. Thus, it follows that the magnitude and direction of **A** are related to its components through the expressions

$$A = \sqrt{A_x^{\,2} + A_y^{\,2}} \tag{3.10}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{3.11}$$

Note that **the signs of the components A_x and A_y depend on the angle θ**. For example, if $\theta = 120°$, then A_x is negative and A_y is positive. If $\theta = 225°$, then both A_x and A_y are negative. Figure 3.14 summarizes the signs of the components when **A** lies in the various quadrants.

When solving problems, you can specify a vector **A** either with its components A_x and A_y or with its magnitude and direction A and θ.

PITFALL PREVENTION

3.2 Component Vectors versus Components

The vectors $\mathbf{A}_x$ and $\mathbf{A}_y$ are the *component vectors* of **A**. These should not be confused with the scalars A_x and A_y, which we shall always refer to as the *components* of **A**.

Quick Quiz 3.5 Choose the correct response to make the sentence true: A component of a vector is (a) always, (b) never, or (c) sometimes larger than the magnitude of the vector.

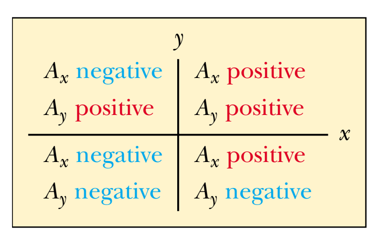


Figure 3.14 The signs of the components of a vector **A** depend on the quadrant in which the vector is located.

Suppose you are working a physics problem that requires resolving a vector into its components. In many applications it is convenient to express the components in a coordinate system having axes that are not horizontal and vertical but are still perpendicular to each other. If you choose reference axes or an angle other than the axes and angle shown in Figure 3.13, the components must be modified accordingly. Suppose a

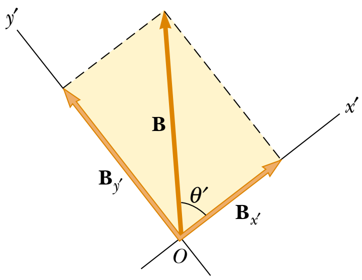


Figure 3.15 The component vectors of **B** in a coordinate system that is tilted.

vector **B** makes an angle θ' with the x' axis defined in Figure 3.15. The components of **B** along the x' and y' axes are $B_{x'} = B\cos\theta'$ and $B_{y'} = B\sin\theta'$, as specified by Equations 3.8 and 3.9. The magnitude and direction of **B** are obtained from expressions equivalent to Equations 3.10 and 3.11. Thus, we can express the components of a vector in any coordinate system that is convenient for a particular situation.

Unit Vectors

Vector quantities often are expressed in terms of unit vectors. **A unit vector is a dimensionless vector having a magnitude of exactly 1.** Unit vectors are used to specify a given direction and have no other physical significance. They are used solely as a convenience in describing a direction in space. We shall use the symbols $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ to represent unit vectors pointing in the positive x, y, and z directions, respectively. (The "hats" on the symbols are a standard notation for unit vectors.) The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ form a set of mutually perpendicular vectors in a right-handed coordinate system, as shown in Figure 3.16a. The magnitude of each unit vector equals 1; that is, $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1$.

Consider a vector **A** lying in the xy plane, as shown in Figure 3.16b. The product of the component A_x and the unit vector $\hat{\mathbf{i}}$ is the vector $A_x\hat{\mathbf{i}}$, which lies on the x axis and has magnitude $|A_x|$. (The vector $A_x\hat{\mathbf{i}}$ is an alternative representation of vector $\mathbf{A}_x$.) Likewise, $A_y\hat{\mathbf{j}}$ is a vector of magnitude $|A_y|$ lying on the y axis. (Again, vector $A_y\hat{\mathbf{j}}$ is an alternative representation of vector $\mathbf{A}_y$.) Thus, the unit–vector notation for the vector **A** is

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \tag{3.12}$$

For example, consider a point lying in the xy plane and having Cartesian coordinates (x, y), as in Figure 3.17. The point can be specified by the **position vector r**, which in unit–vector form is given by

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \tag{3.13}$$

This notation tells us that the components of **r** are the lengths x and y.

Now let us see how to use components to add vectors when the graphical method is not sufficiently accurate. Suppose we wish to add vector **B** to vector **A** in Equation 3.12, where vector **B** has components B_x and B_y. All we do is add the x and y components separately. The resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$ is therefore

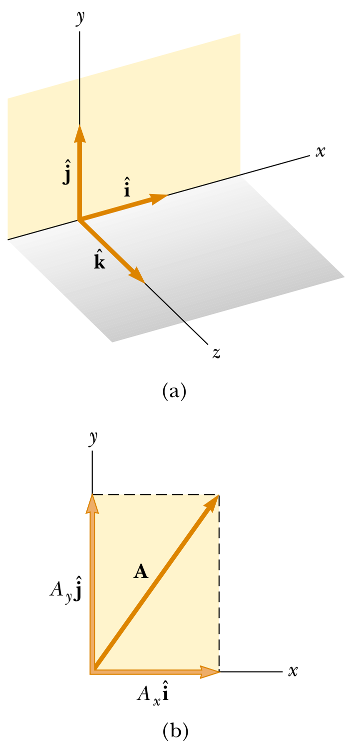
$$\mathbf{R} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}})$$

or

$$\mathbf{R} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} \tag{3.14}$$

Because $\mathbf{R} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$, we see that the components of the resultant vector are

$$\begin{aligned} R_x &= A_x + B_x \\ R_y &= A_y + B_y \end{aligned} \tag{3.15}$$



Active Figure 3.16 (a) The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are directed along the x, y, and z axes, respectively. (b) Vector $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$ lying in the xy plane has components A_x and A_y.

At the Active Figures link at http://www.pse6.com you can rotate the coordinate axes in 3-dimensional space and view a representation of vector A in three dimensions.

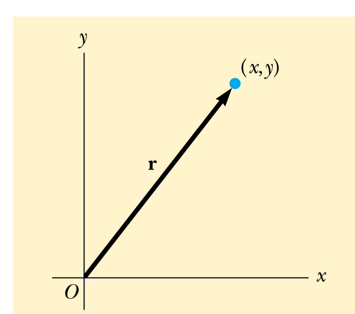


Figure 3.17 The point whose Cartesian coordinates are (x, y) can be represented by the position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

We obtain the magnitude of **R** and the angle it makes with the x axis from its components, using the relationships

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \tag{3.16}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x} \tag{3.17}$$

We can check this addition by components with a geometric construction, as shown in Figure 3.18. Remember that you must note the signs of the components when using either the algebraic or the graphical method.

At times, we need to consider situations involving motion in three component directions. The extension of our methods to three-dimensional vectors is straightforward. If **A** and **B** both have x, y, and z components, we express them in the form

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{3.18}$$

$$\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{3.19}$$

The sum of **A** and **B** is

$$\mathbf{R} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}} \tag{3.20}$$

Note that Equation 3.20 differs from Equation 3.14: in Equation 3.20, the resultant vector also has a z component $R_z = A_z + B_z$. If a vector **R** has x, y, and z components, the magnitude of the vector is $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$. The angle θ_x that **R** makes with the x axis is found from the expression $\cos\theta_x = R_x/R$, with similar expressions for the angles with respect to the y and z axes.

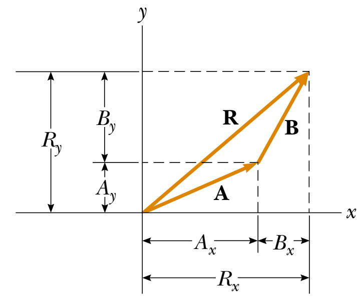


Figure 3.18 This geometric construction for the sum of two vectors shows the relationship between the components of the resultant **R** and the components of the individual vectors.

Quick Quiz 3.6 If at least one component of a vector is a positive number, the vector cannot (a) have any component that is negative (b) be zero (c) have three dimensions.

Quick Quiz 3.7 If $\mathbf{A} + \mathbf{B} = 0$, the corresponding components of the two vectors **A** and **B** must be (a) equal (b) positive (c) negative (d) of opposite sign.

Quick Quiz 3.8 For which of the following vectors is the magnitude of the vector equal to one of the components of the vector? (a) $\mathbf{A} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (b) $\mathbf{B} = -3\hat{\mathbf{j}}$ (c) $\mathbf{C} = +5\hat{\mathbf{k}}$

PROBLEM-SOLVING HINTS

Adding Vectors

When you need to add two or more vectors, use this step-by-step procedure:

- Select a coordinate system that is convenient. (Try to reduce the number of components you need to calculate by choosing axes that line up with as many vectors as possible.)
- Draw a labeled sketch of the vectors described in the problem.
- Find the x and y components of all vectors and the resultant components (the algebraic sum of the components) in the x and y directions.
- If necessary, use the Pythagorean theorem to find the magnitude of the resultant vector and select a suitable trigonometric function to find the angle that the resultant vector makes with the x axis.

PITFALL PREVENTION

3.3 *x* and *y* Components

Equations 3.8 and 3.9 associate the cosine of the angle with the x component and the sine of the angle with the y component. This is true *only* because we measured the angle θ with respect to the x axis, so don't memorize these equations. If θ is measured with respect to the y axis (as in some problems), these equations will be incorrect. Think about which side of the triangle containing the components is adjacent to the angle and which side is opposite, and assign the cosine and sine accordingly.

PITFALL PREVENTION

3.4 Tangents on Calculators

Generally, the inverse tangent function on calculators provides an angle between $-90°$ and $+90°$. As a consequence, if the vector you are studying lies in the second or third quadrant, the angle measured from the positive x axis will be the angle your calculator returns plus 180°.

Example 3.3 The Sum of Two Vectors

Find the sum of two vectors **A** and **B** lying in the *xy* plane and given by

$$\mathbf{A} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\ \text{m} \qquad \text{and} \qquad \mathbf{B} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})\ \text{m}$$

Solution You may wish to draw the vectors to *conceptualize* the situation. We *categorize* this as a simple plug-in problem. Comparing this expression for **A** with the general expression $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$, we see that $A_x = 2.0$ m and $A_y = 2.0$ m. Likewise, $B_x = 2.0$ m and $B_y = -4.0$ m. We obtain the resultant vector **R**, using Equation 3.14:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (2.0 + 2.0)\hat{\mathbf{i}}\ \text{m} + (2.0 - 4.0)\hat{\mathbf{j}}\ \text{m}$$
$$= (4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}})\ \text{m}$$

or

$$R_x = 4.0\ \text{m} \qquad R_y = -2.0\ \text{m}$$

The magnitude of **R** is found using Equation 3.16:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0\ \text{m})^2 + (-2.0\ \text{m})^2} = \sqrt{20}\ \text{m}$$
$$= 4.5\ \text{m}$$

We can find the direction of **R** from Equation 3.17:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0\ \text{m}}{4.0\ \text{m}} = -0.50$$

Your calculator likely gives the answer $-27°$ for $\theta = \tan^{-1}(-0.50)$. This answer is correct if we interpret it to mean 27° clockwise from the *x* axis. Our standard form has been to quote the angles measured counterclockwise from the $+x$ axis, and that angle for this vector is $\theta = 333°$.

Example 3.4 The Resultant Displacement

A particle undergoes three consecutive displacements: $\mathbf{d}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})$ cm, $\mathbf{d}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})$ cm and $\mathbf{d}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})$ cm. Find the components of the resultant displacement and its magnitude.

Solution Three-dimensional displacements are more difficult to imagine than those in two dimensions, because the latter can be drawn on paper. For this problem, let us *conceptualize* that you start with your pencil at the origin of a piece of graph paper on which you have drawn *x* and *y* axes. Move your pencil 15 cm to the right along the *x* axis, then 30 cm upward along the *y* axis, and then 12 cm *vertically away* from the graph paper. This provides the displacement described by $\mathbf{d}_1$. From this point, move your pencil 23 cm to the right parallel to the *x* axis, 14 cm parallel to the graph paper in the $-y$ direction, and then 5.0 cm vertically downward toward the graph paper. You are now at the displacement from the origin described by $\mathbf{d}_1 + \mathbf{d}_2$. From this point, move your pencil 13 cm to the left in the $-x$ direction, and (finally!) 15 cm parallel to the graph paper along the *y* axis. Your final position is at a displacement $\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$ from the origin.

Despite the difficulty in conceptualizing in three dimensions, we can *categorize* this problem as a plug-in problem due to the careful bookkeeping methods that we have developed for vectors. The mathematical manipulation keeps track of this motion along the three perpendicular axes in an organized, compact way:

$$\mathbf{R} = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$$
$$= (15 + 23 - 13)\hat{\mathbf{i}}\ \text{cm} + (30 - 14 + 15)\hat{\mathbf{j}}\ \text{cm} + (12 - 5.0 + 0)\hat{\mathbf{k}}\ \text{cm}$$
$$= (25\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 7.0\hat{\mathbf{k}})\ \text{cm}$$

The resultant displacement has components $R_x = 25$ cm, $R_y = 31$ cm, and $R_z = 7.0$ cm. Its magnitude is

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$
$$= \sqrt{(25\ \text{cm})^2 + (31\ \text{cm})^2 + (7.0\ \text{cm})^2} = 40\ \text{cm}$$

Example 3.5 Taking a Hike

Interactive

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.

Solution We *conceptualize* the problem by drawing a sketch as in Figure 3.19. If we denote the displacement vectors on the first and second days by **A** and **B**, respectively, and use the car as the origin of coordinates, we obtain the vectors shown in Figure 3.19. Drawing the resultant **R**, we can now *categorize* this as a problem we've solved before—an addition of two vectors. This should give you a hint of the power of categorization—many new problems are very similar to problems that we have already solved if we are careful to conceptualize them.

We will *analyze* this problem by using our new knowledge of vector components. Displacement **A** has a magnitude of 25.0 km and is directed 45.0° below the positive *x* axis. From Equations 3.8 and 3.9, its components are

$$A_x = A\cos(-45.0°) = (25.0\ \text{km})(0.707) = 17.7\ \text{km}$$

$$A_y = A\sin(-45.0°) = (25.0\ \text{km})(-0.707) = -17.7\ \text{km}$$

The negative value of A_y indicates that the hiker walks in the negative *y* direction on the first day. The signs of A_x and A_y also are evident from Figure 3.19.

The second displacement **B** has a magnitude of 40.0 km and is 60.0° north of east. Its components are

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = \boxed{20.0\text{ km}}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = \boxed{34.6\text{ km}}$$

(B) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.

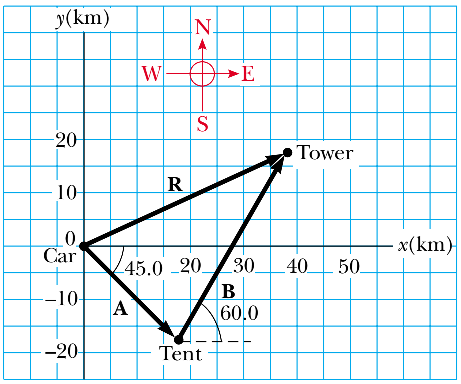


Figure 3.19 (Example 3.5) The total displacement of the hiker is the vector **R** = **A** + **B.**

Solution The resultant displacement for the trip **R** = **A** + **B** has components given by Equation 3.15:

$$R_x = A_x + B_x = 17.7\text{ km} + 20.0\text{ km} = \boxed{37.7\text{ km}}$$

$$R_y = A_y + B_y = -17.7\text{ km} + 34.6\text{ km} = \boxed{16.9\text{ km}}$$

In unit–vector form, we can write the total displacement as

$$\mathbf{R} = \boxed{(37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}})\text{ km}}$$

Using Equations 3.16 and 3.17, we find that the vector **R** has a magnitude of 41.3 km and is directed 24.1° north of east.

Let us *finalize*. The units of **R** are km, which is reasonable for a displacement. Looking at the graphical representation in Figure 3.19, we estimate that the final position of the hiker is at about (38 km, 17 km) which is consistent with the components of **R** in our final result. Also, both components of **R** are positive, putting the final position in the first quadrant of the coordinate system, which is also consistent with Figure 3.19.

Investigate this situation at the Interactive Worked Example link at **http://www.pse6.com.**

Example 3.6 Let's Fly Away!

A commuter airplane takes the route shown in Figure 3.20. First, it flies from the origin of the coordinate system shown to city A, located 175 km in a direction 30.0° north of east. Next, it flies 153 km 20.0° west of north to city B. Finally, it flies 195 km due west to city C. Find the location of city C relative to the origin.

Solution Once again, a drawing such as Figure 3.20 allows us to *conceptualize* the problem. It is convenient to choose the coordinate system shown in Figure 3.20, where the *x* axis points to the east and the *y* axis points to the north. Let us denote the three consecutive displacements by the vectors **a**, **b**, and **c**.

We can now *categorize* this problem as being similar to Example 3.5 that we have already solved. There are two primary differences. First, we are adding three vectors instead of two. Second, Example 3.5 guided us by first asking for the components in part (A). The current Example has no such guidance and simply asks for a result. We need to *analyze* the situation and choose a path. We will follow the same pattern that we did in Example 3.5, beginning with finding the components of the three vectors **a**, **b**, and **c**. Displacement **a** has a magnitude of 175 km and the components

$$a_x = a\cos(30.0°) = (175\text{ km})(0.866) = 152\text{ km}$$

$$a_y = a\sin(30.0°) = (175\text{ km})(0.500) = 87.5\text{ km}$$

Displacement **b**, whose magnitude is 153 km, has the components

$$b_x = b\cos(110°) = (153\text{ km})(-0.342) = -52.3\text{ km}$$

$$b_y = b\sin(110°) = (153\text{ km})(0.940) = 144\text{ km}$$

Finally, displacement **c,** whose magnitude is 195 km, has the components

$$c_x = c\cos(180°) = (195\text{ km})(-1) = -195\text{ km}$$

$$c_y = c\sin(180°) = 0$$

Therefore, the components of the position vector **R** from the starting point to city C are

$$R_x = a_x + b_x + c_x = 152\text{ km} - 52.3\text{ km} - 195\text{ km}$$

$$= \boxed{-95.3\text{ km}}$$

$$R_y = a_y + b_y + c_y = 87.5\text{ km} + 144\text{ km} + 0 = \boxed{232\text{ km}}$$

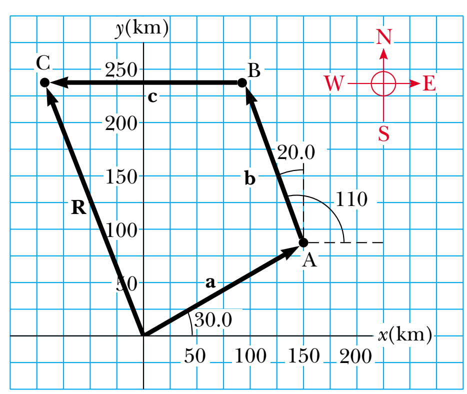


Figure 3.20 (Example 3.6) The airplane starts at the origin, flies first to city A, then to city B, and finally to city C.

In unit–vector notation, $\mathbf{R} = (-95.3\hat{\mathbf{i}} + 232\hat{\mathbf{j}})$ km. Using Equations 3.16 and 3.17, we find that the vector **R** has a magnitude of 251 km and is directed 22.3° west of north.

To *finalize* the problem, note that the airplane can reach city C from the starting point by first traveling 95.3 km due west and then by traveling 232 km due north. Or it could follow a straight-line path to C by flying a distance $R =$ 251 km in a direction 22.3° west of north.

What If? **After landing in city C, the pilot wishes to return to the origin along a single straight line. What are the components of the vector representing this displacement? What should the heading of the plane be?**

Answer The desired vector **H** (for Home!) is simply the negative of vector **R**:

$$\mathbf{H} = -\mathbf{R} = (+95.3\hat{\mathbf{i}} - 232\hat{\mathbf{j}}) \text{ km}$$

The heading is found by calculating the angle that the vector makes with the x axis:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-232 \text{ m}}{95.3 \text{ m}} = -2.43$$

This gives a heading angle of $\theta = -67.7°$, or 67.7° south of east.

Take a practice test for this chapter by clicking on the Practice Test link at http://www.pse6.com.

SUMMARY

Scalar quantities are those that have only a numerical value and no associated direction. **Vector quantities** have both magnitude and direction and obey the laws of vector addition. The magnitude of a vector is *always* a positive number.

When two or more vectors are added together, all of them must have the same units and all of them must be the same type of quantity. We can add two vectors **A** and **B** graphically. In this method (Fig. 3.6), the resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$ runs from the tail of **A** to the tip of **B**.

A second method of adding vectors involves **components** of the vectors. The x component A_x of the vector **A** is equal to the projection of **A** along the x axis of a coordinate system, as shown in Figure 3.13, where $A_x = A\cos\theta$. The y component A_y of **A** is the projection of **A** along the y axis, where $A_y = A\sin\theta$. Be sure you can determine which trigonometric functions you should use in all situations, especially when θ is defined as something other than the counterclockwise angle from the positive x axis.

If a vector **A** has an x component A_x and a y component A_y, the vector can be expressed in unit–vector form as $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$. In this notation, $\hat{\mathbf{i}}$ is a unit vector pointing in the positive x direction, and $\hat{\mathbf{j}}$ is a unit vector pointing in the positive y direction. Because $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors, $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = 1$.

We can find the resultant of two or more vectors by resolving all vectors into their x and y components, adding their resultant x and y components, and then using the Pythagorean theorem to find the magnitude of the resultant vector. We can find the angle that the resultant vector makes with respect to the x axis by using a suitable trigonometric function.

QUESTIONS

1. Two vectors have unequal magnitudes. Can their sum be zero? Explain.

2. Can the magnitude of a particle's displacement be greater than the distance traveled? Explain.

3. The magnitudes of two vectors **A** and **B** are $A = 5$ units and $B = 2$ units. Find the largest and smallest values possible for the magnitude of the resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$.

4. Which of the following are vectors and which are not: force, temperature, the volume of water in a can, the ratings of a TV show, the height of a building, the velocity of a sports car, the age of the Universe?

$\boxed{\textbf{5.}}$ A vector **A** lies in the xy plane. For what orientations of **A** will both of its components be negative? For what orientations will its components have opposite signs?

6. A book is moved once around the perimeter of a tabletop with the dimensions 1.0 m × 2.0 m. If the book ends up at its initial position, what is its displacement? What is the distance traveled?

7. While traveling along a straight interstate highway you notice that the mile marker reads 260. You travel until you reach mile marker 150 and then retrace your path to the mile marker 175. What is the magnitude of your resultant displacement from mile marker 260?

8. If the component of vector **A** along the direction of vector **B** is zero, what can you conclude about the two vectors?

9. Can the magnitude of a vector have a negative value? Explain.

10. Under what circumstances would a nonzero vector lying in the *xy* plane have components that are equal in magnitude?

11. If **A** = **B**, what can you conclude about the components of **A** and **B**?

12. Is it possible to add a vector quantity to a scalar quantity? Explain.

13. The resolution of vectors into components is equivalent to replacing the original vector with the sum of two vectors, whose sum is the same as the original vector. There are an infinite number of pairs of vectors that will satisfy this condition; we choose that pair with one vector parallel to the *x* axis and the second parallel to the *y* axis. What difficulties would be introduced by defining components relative to axes that are not perpendicular—for example, the *x* axis and a *y* axis oriented at 45° to the *x* axis?

14. In what circumstance is the *x* component of a vector given by the magnitude of the vector times the sine of its direction angle?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*

= coached solution with hints available at http://www.pse6.com = computer useful in solving problem

= paired numerical and symbolic problems

Section 3.1 Coordinate Systems

1. The polar coordinates of a point are $r = 5.50$ m and $\theta = 240°$. What are the Cartesian coordinates of this point?

2. Two points in a plane have polar coordinates (2.50 m, 30.0°) and (3.80 m, 120.0°). Determine (a) the Cartesian coordinates of these points and (b) the distance between them.

3. A fly lands on one wall of a room. The lower left-hand corner of the wall is selected as the origin of a two-dimensional Cartesian coordinate system. If the fly is located at the point having coordinates (2.00, 1.00) m, (a) how far is it from the corner of the room? (b) What is its location in polar coordinates?

4. Two points in the *xy* plane have Cartesian coordinates (2.00, − 4.00) m and (− 3.00, 3.00) m. Determine (a) the distance between these points and (b) their polar coordinates.

5. If the rectangular coordinates of a point are given by (2, y) and its polar coordinates are (r, 30°), determine y and r.

6. If the polar coordinates of the point (x, y) are (r, θ), determine the polar coordinates for the points: (a) $(-x, y)$, (b) $(-2x, -2y)$, and (c) $(3x, -3y)$.

Section 3.2 Vector and Scalar Quantities

Section 3.3 Some Properties of Vectors

7. A surveyor measures the distance across a straight river by the following method: starting directly across from a tree on the opposite bank, she walks 100 m along the riverbank to establish a baseline. Then she sights across to the tree. The angle from her baseline to the tree is 35.0°. How wide is the river?

8. A pedestrian moves 6.00 km east and then 13.0 km north. Find the magnitude and direction of the resultant displacement vector using the graphical method.

9. A plane flies from base camp to lake A, 280 km away, in a direction of 20.0° north of east. After dropping off supplies it flies to lake B, which is 190 km at 30.0° west of north from lake A. Graphically determine the distance and direction from lake B to the base camp.

10. Vector **A** has a magnitude of 8.00 units and makes an angle of 45.0° with the positive *x* axis. Vector **B** also has a magnitude of 8.00 units and is directed along the negative *x* axis. Using graphical methods, find (a) the vector sum **A** + **B** and (b) the vector difference **A** − **B**.

11. A skater glides along a circular path of radius 5.00 m. If he coasts around one half of the circle, find (a) the magnitude of the displacement vector and (b) how far the person skated. (c) What is the magnitude of the displacement if he skates all the way around the circle?

12. A force $\mathbf{F}_1$ of magnitude 6.00 units acts at the origin in a direction 30.0° above the positive *x* axis. A second force $\mathbf{F}_2$ of magnitude 5.00 units acts at the origin in the direction of the positive *y* axis. Find graphically the magnitude and direction of the resultant force $\mathbf{F}_1 + \mathbf{F}_2$.

13. Arbitrarily define the "instantaneous vector height" of a person as the displacement vector from the point halfway

between his or her feet to the top of the head. Make an order-of-magnitude estimate of the total vector height of all the people in a city of population 100 000 (a) at 10 o'clock on a Tuesday morning, and (b) at 5 o'clock on a Saturday morning. Explain your reasoning.

14. A dog searching for a bone walks 3.50 m south, then runs 8.20 m at an angle 30.0° north of east, and finally walks 15.0 m west. Find the dog's resultant displacement vector using graphical techniques.

15. Each of the displacement vectors **A** and **B** shown in Fig. P3.15 has a magnitude of 3.00 m. Find graphically (a) **A** + **B**, (b) **A** − **B**, (c) **B** − **A**, (d) **A** − 2**B**. Report all angles counterclockwise from the positive *x* axis.

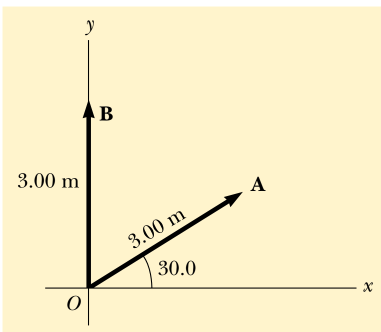


Figure P3.15 Problems 15 and 37.

16. Three displacements are **A** = 200 m, due south; **B** = 250 m, due west; **C** = 150 m, 30.0° east of north. Construct a separate diagram for each of the following possible ways of adding these vectors: $\mathbf{R}_1 = \mathbf{A} + \mathbf{B} + \mathbf{C}$; $\mathbf{R}_2 = \mathbf{B} + \mathbf{C} + \mathbf{A}$; $\mathbf{R}_3 = \mathbf{C} + \mathbf{B} + \mathbf{A}$.

17. A roller coaster car moves 200 ft horizontally, and then rises 135 ft at an angle of 30.0° above the horizontal. It then travels 135 ft at an angle of 40.0° downward. What is its displacement from its starting point? Use graphical techniques.

Section 3.4 Components of a Vector and Unit Vectors

18. Find the horizontal and vertical components of the 100-m displacement of a superhero who flies from the top of a tall building following the path shown in Fig. P3.18.

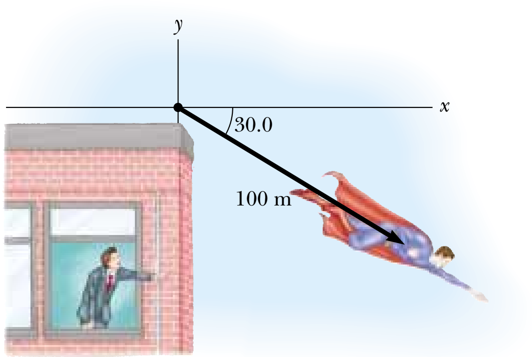


Figure P3.18

19. A vector has an *x* component of −25.0 units and a *y* component of 40.0 units. Find the magnitude and direction of this vector.

20. A person walks 25.0° north of east for 3.10 km. How far would she have to walk due north and due east to arrive at the same location?

21. Obtain expressions in component form for the position vectors having the following polar coordinates: (a) 12.8 m, 150° (b) 3.30 cm, 60.0° (c) 22.0 in., 215°.

22. A displacement vector lying in the *xy* plane has a magnitude of 50.0 m and is directed at an angle of 120° to the positive *x* axis. What are the rectangular components of this vector?

23. A girl delivering newspapers covers her route by traveling 3.00 blocks west, 4.00 blocks north, and then 6.00 blocks east. (a) What is her resultant displacement? (b) What is the total distance she travels?

24. In 1992, Akira Matsushima, from Japan, rode a unicycle across the United States, covering about 4 800 km in six weeks. Suppose that, during that trip, he had to find his way through a city with plenty of one-way streets. In the city center, Matsushima had to travel in sequence 280 m north, 220 m east, 360 m north, 300 m west, 120 m south, 60.0 m east, 40.0 m south, 90.0 m west (road construction) and then 70.0 m north. At that point, he stopped to rest. Meanwhile, a curious crow decided to fly the distance from his starting point to the rest location directly ("as the crow flies"). It took the crow 40.0 s to cover that distance. Assuming the velocity of the crow was constant, find its magnitude and direction.

25. While exploring a cave, a spelunker starts at the entrance and moves the following distances. She goes 75.0 m north, 250 m east, 125 m at an angle 30.0° north of east, and 150 m south. Find the resultant displacement from the cave entrance.

26. A map suggests that Atlanta is 730 miles in a direction of 5.00° north of east from Dallas. The same map shows that Chicago is 560 miles in a direction of 21.0° west of north from Atlanta. Modeling the Earth as flat, use this information to find the displacement from Dallas to Chicago.

27. Given the vectors $\mathbf{A} = 2.00\hat{\mathbf{i}} + 6.00\hat{\mathbf{j}}$ and $\mathbf{B} = 3.00\hat{\mathbf{i}} - 2.00\hat{\mathbf{j}}$, (a) draw the vector sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and the vector difference $\mathbf{D} = \mathbf{A} - \mathbf{B}$. (b) Calculate **C** and **D**, first in terms of unit vectors and then in terms of polar coordinates, with angles measured with respect to the + *x* axis.

28. Find the magnitude and direction of the resultant of three displacements having rectangular components (3.00, 2.00) m, (−5.00, 3.00) m, and (6.00, 1.00) m.

29. A man pushing a mop across a floor causes it to undergo two displacements. The first has a magnitude of 150 cm and makes an angle of 120° with the positive *x* axis. The resultant displacement has a magnitude of 140 cm and is directed at an angle of 35.0° to the positive *x* axis. Find the magnitude and direction of the second displacement.

30. Vector **A** has *x* and *y* components of −8.70 cm and 15.0 cm, respectively; vector **B** has *x* and *y* components of 13.2 cm and −6.60 cm, respectively. If $\mathbf{A} - \mathbf{B} + 3\mathbf{C} = 0$, what are the components of **C**?

31. Consider the two vectors $\mathbf{A} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $\mathbf{B} = -\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$. Calculate (a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$, (c) $|\mathbf{A} + \mathbf{B}|$, (d) $|\mathbf{A} - \mathbf{B}|$, and (e) the directions of $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$.

32. Consider the three displacement vectors $\mathbf{A} = (3\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$ m, $\mathbf{B} = (\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$ m, and $\mathbf{C} = (-2\hat{\mathbf{i}} + 5\hat{\mathbf{j}})$ m. Use the component method to determine (a) the magnitude and direction of the vector $\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C}$, (b) the magnitude and direction of $\mathbf{E} = -\mathbf{A} - \mathbf{B} + \mathbf{C}$.

33. A particle undergoes the following consecutive displacements: 3.50 m south, 8.20 m northeast, and 15.0 m west. What is the resultant displacement?

34. In a game of American football, a quarterback takes the ball from the line of scrimmage, runs backward a distance of 10.0 yards, and then sideways parallel to the line of scrimmage for 15.0 yards. At this point, he throws a forward pass 50.0 yards straight downfield perpendicular to the line of scrimmage. What is the magnitude of the football's resultant displacement?

35. The helicopter view in Fig. P3.35 shows two people pulling on a stubborn mule. Find (a) the single force that is equivalent to the two forces shown, and (b) the force that a third person would have to exert on the mule to make the resultant force equal to zero. The forces are measured in units of newtons (abbreviated N).

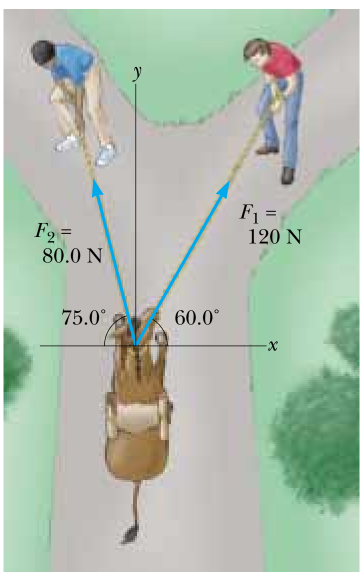


Figure P3.35

36. A novice golfer on the green takes three strokes to sink the ball. The successive displacements are 4.00 m to the north, 2.00 m northeast, and 1.00 m at 30.0° west of south. Starting at the same initial point, an expert golfer could make the hole in what single displacement?

37. Use the component method to add the vectors **A** and **B** shown in Figure P3.15. Express the resultant $\mathbf{A} + \mathbf{B}$ in unit–vector notation.

38. In an assembly operation illustrated in Figure P3.38, a robot moves an object first straight upward and then also to the east, around an arc forming one quarter of a circle of radius 4.80 cm that lies in an east-west vertical plane. The robot then moves the object upward and to the north, through a quarter of a circle of radius 3.70 cm that lies in a north-south vertical plane. Find (a) the magnitude of the total displacement of the object, and (b) the angle the total displacement makes with the vertical.

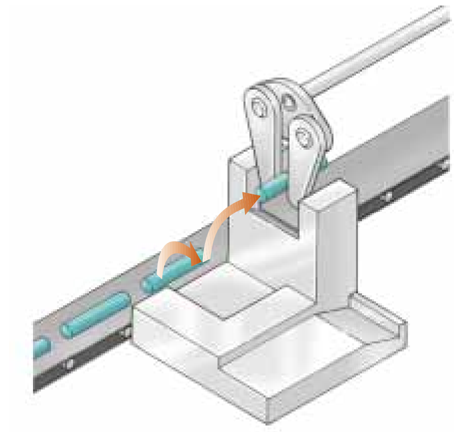


Figure P3.38

39. Vector **B** has *x*, *y*, and *z* components of 4.00, 6.00, and 3.00 units, respectively. Calculate the magnitude of **B** and the angles that **B** makes with the coordinate axes.

40. You are standing on the ground at the origin of a coordinate system. An airplane flies over you with constant velocity parallel to the *x* axis and at a fixed height of 7.60×10^3 m. At time $t = 0$ the airplane is directly above you, so that the vector leading from you to it is $\mathbf{P}_0 = (7.60 \times 10^3\ \text{m})\hat{\mathbf{j}}$. At $t = 30.0$ s the position vector leading from you to the airplane is $\mathbf{P}_{30} = (8.04 \times 10^3\ \text{m})\hat{\mathbf{i}} + (7.60 \times 10^3\ \text{m})\hat{\mathbf{j}}$. Determine the magnitude and orientation of the airplane's position vector at $t = 45.0$ s.

41. The vector **A** has *x*, *y*, and *z* components of 8.00, 12.0, and −4.00 units, respectively. (a) Write a vector expression for **A** in unit–vector notation. (b) Obtain a unit–vector expression for a vector **B** one fourth the length of **A** pointing in the same direction as **A**. (c) Obtain a unit–vector expression for a vector **C** three times the length of **A** pointing in the direction opposite the direction of **A**.

42. Instructions for finding a buried treasure include the following: Go 75.0 paces at 240°, turn to 135° and walk 125 paces, then travel 100 paces at 160°. The angles are measured counterclockwise from an axis pointing to the east, the $+x$ direction. Determine the resultant displacement from the starting point.

43. Given the displacement vectors $\mathbf{A} = (3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ m and $\mathbf{B} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 7\hat{\mathbf{k}})$ m, find the magnitudes of the vectors (a) $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and (b) $\mathbf{D} = 2\mathbf{A} - \mathbf{B}$, also expressing each in terms of its rectangular components.

44. A radar station locates a sinking ship at range 17.3 km and bearing 136° clockwise from north. From the same station a rescue plane is at horizontal range 19.6 km, 153° clockwise from north, with elevation 2.20 km. (a) Write the position vector for the ship relative to the plane, letting $\hat{\mathbf{i}}$ represent east, $\hat{\mathbf{j}}$ north, and $\hat{\mathbf{k}}$ up. (b) How far apart are the plane and ship?

45. As it passes over Grand Bahama Island, the eye of a hurricane is moving in a direction 60.0° north of west with a speed of 41.0 km/h. Three hours later, the course of the hurricane suddenly shifts due north, and its speed slows to 25.0 km/h. How far from Grand Bahama is the eye 4.50 h after it passes over the island?

46. (a) Vector **E** has magnitude 17.0 cm and is directed 27.0° counterclockwise from the $+x$ axis. Express it in unit–vector notation. (b) Vector **F** has magnitude 17.0 cm and is directed 27.0° counterclockwise from the $+y$ axis. Express it in unit–vector notation. (c) Vector **G** has magnitude 17.0 cm and is directed 27.0° clockwise from the $-y$ axis. Express it in unit–vector notation.

47. Vector **A** has a negative x component 3.00 units in length and a positive y component 2.00 units in length. (a) Determine an expression for **A** in unit–vector notation. (b) Determine the magnitude and direction of **A**. (c) What vector **B** when added to **A** gives a resultant vector with no x component and a negative y component 4.00 units in length?

48. An airplane starting from airport A flies 300 km east, then 350 km at 30.0° west of north, and then 150 km north to arrive finally at airport B. (a) The next day, another plane flies directly from A to B in a straight line. In what direction should the pilot travel in this direct flight? (b) How far will the pilot travel in this direct flight? Assume there is no wind during these flights.

49. Three displacement vectors of a croquet ball are shown in Figure P3.49, where $|\mathbf{A}| = 20.0$ units, $|\mathbf{B}| = 40.0$ units, and $|\mathbf{C}| = 30.0$ units. Find (a) the resultant in unit–vector notation and (b) the magnitude and direction of the resultant displacement.

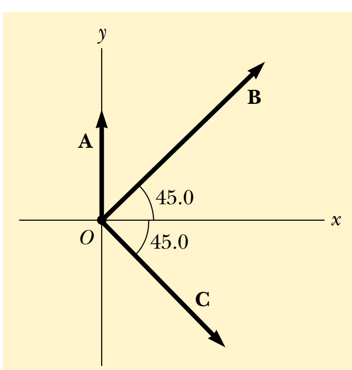


Figure P3.49

50. If $\mathbf{A} = (6.00\hat{\mathbf{i}} - 8.00\hat{\mathbf{j}})$ units, $\mathbf{B} = (-8.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})$ units, and $\mathbf{C} = (26.0\hat{\mathbf{i}} + 19.0\hat{\mathbf{j}})$ units, determine a and b such that $a\mathbf{A} + b\mathbf{B} + \mathbf{C} = 0$.

Additional Problems

51. Two vectors **A** and **B** have precisely equal magnitudes. In order for the magnitude of **A** + **B** to be one hundred times larger than the magnitude of **A** − **B**, what must be the angle between them?

52. Two vectors **A** and **B** have precisely equal magnitudes. In order for the magnitude of **A** + **B** to be larger than the magnitude of **A** − **B** by the factor n, what must be the angle between them?

53. A vector is given by $\mathbf{R} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Find (a) the magnitudes of the x, y, and z components, (b) the magnitude of **R**, and (c) the angles between **R** and the x, y, and z axes.

54. The biggest stuffed animal in the world is a snake 420 m long, constructed by Norwegian children. Suppose the snake is laid out in a park as shown in Figure P3.54, forming two straight sides of a 105° angle, with one side 240 m long. Olaf and Inge run a race they invent. Inge runs directly from the tail of the snake to its head and Olaf starts from the same place at the same time but runs along the snake. If both children run steadily at 12.0 km/h, Inge reaches the head of the snake how much earlier than Olaf?

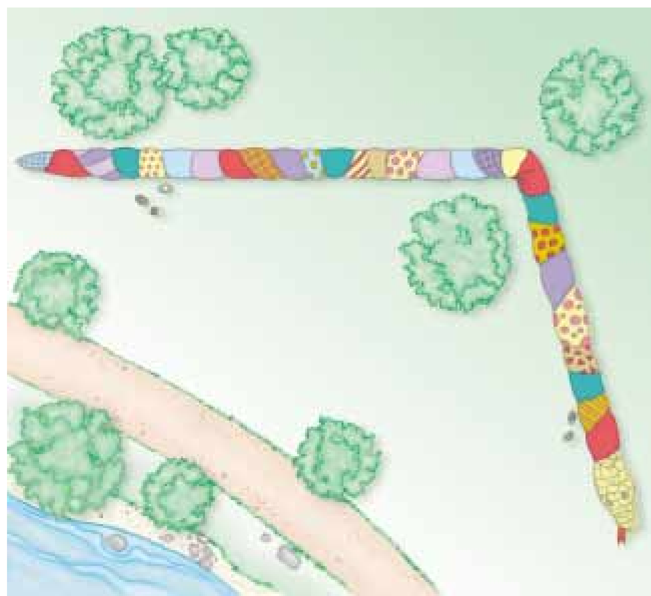


Figure P3.54

55. An air-traffic controller observes two aircraft on his radar screen. The first is at altitude 800 m, horizontal distance 19.2 km, and 25.0° south of west. The second aircraft is at altitude 1 100 m, horizontal distance 17.6 km, and 20.0° south of west. What is the distance between the two aircraft? (Place the x axis west, the y axis south, and the z axis vertical.)

56. A ferry boat transports tourists among three islands. It sails from the first island to the second island, 4.76 km away, in a direction 37.0° north of east. It then sails from the second island to the third island in a direction 69.0° west of north. Finally it returns to the first island, sailing in a direction 28.0° east of south. Calculate the distance between (a) the second and third islands (b) the first and third islands.

57. The rectangle shown in Figure P3.57 has sides parallel to the x and y axes. The position vectors of two corners are $\mathbf{A} = 10.0$ m at 50.0° and $\mathbf{B} = 12.0$ m at 30.0°. (a) Find the perimeter of the rectangle. (b) Find the magnitude and direction of the vector from the origin to the upper right corner of the rectangle.

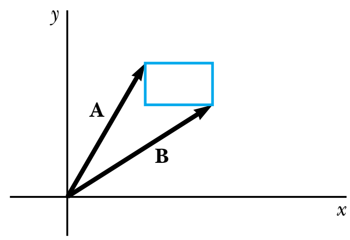


Figure P3.57

58. Find the sum of these four vector forces: 12.0 N to the right at 35.0° above the horizontal, 31.0 N to the left at 55.0° above the horizontal, 8.40 N to the left at 35.0° below the horizontal, and 24.0 N to the right at 55.0° below the horizontal. Follow these steps: Make a drawing of this situation and select the best axes for x and y so you have the least number of components. Then add the vectors by the component method.

59. A person going for a walk follows the path shown in Fig. P3.59. The total trip consists of four straight-line paths. At the end of the walk, what is the person's resultant displacement measured from the starting point?

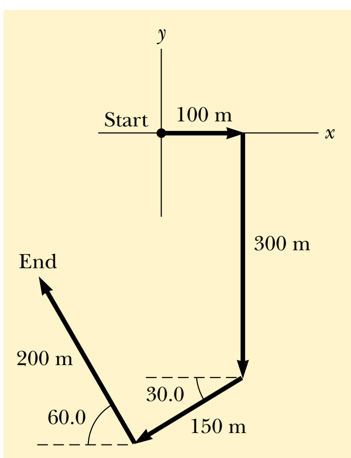


Figure P3.59

60. The instantaneous position of an object is specified by its position vector $\mathbf{r}$ leading from a fixed origin to the location of the point object. Suppose that for a certain object the position vector is a function of time, given by $\mathbf{r} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2t\hat{\mathbf{k}}$, where r is in meters and t is in seconds. Evaluate $d\mathbf{r}/dt$. What does it represent about the object?

61. A jet airliner, moving initially at 300 mi/h to the east, suddenly enters a region where the wind is blowing at 100 mi/h toward the direction 30.0° north of east. What are the new speed and direction of the aircraft relative to the ground?

62. Long John Silver, a pirate, has buried his treasure on an island with five trees, located at the following points: (30.0 m, − 20.0 m), (60.0 m, 80.0 m), (− 10.0 m, − 10.0 m), (40.0 m, − 30.0 m), and (− 70.0 m, 60.0 m), all measured relative to some origin, as in Figure P3.62. His ship's log instructs you to start at tree A and move toward tree B, but to cover only one half the distance between A and B. Then move toward tree C, covering one third the distance between your current location and C. Next move toward D, covering one fourth the distance between where you are and D. Finally move towards E, covering one fifth the distance between you and E, stop, and dig. (a) Assume that you have correctly determined the order in which the pirate labeled the trees as A, B, C, D, and E, as shown in the figure. What are the coordinates of the point where his treasure is buried? (b) **What if** you do not really know the way the pirate labeled the trees? Rearrange the order of the trees [for instance, B(30 m, − 20 m), A(60 m, 80 m), E(− 10 m, − 10 m), C(40 m, − 30 m), and D(− 70 m, 60 m)] and repeat the calculation to show that the answer does not depend on the order in which the trees are labeled.

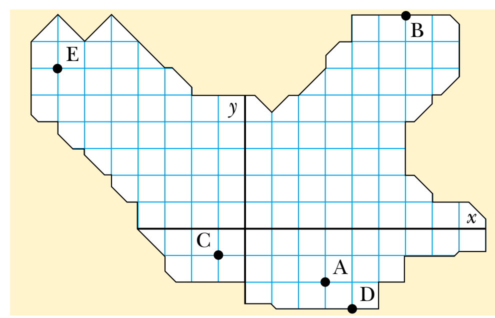


Figure P3.62

63. Consider a game in which N children position themselves at equal distances around the circumference of a circle. At the center of the circle is a rubber tire. Each child holds a rope attached to the tire and, at a signal, pulls on his rope. All children exert forces of the same magnitude F. In the case $N = 2$, it is easy to see that the net force on the tire will be zero, because the two oppositely directed force vectors add to zero. Similarly, if $N = 4$, 6, or any even integer, the resultant force on the tire must be zero, because the forces exerted by each pair of oppositely positioned children will cancel. When an odd number of children are around the circle, it is not so obvious whether the total force on the central tire will be zero. (a) Calculate the net force on the tire in the case $N = 3$, by adding the components of the three force vectors. Choose the x axis to lie along one of the ropes. (b) **What If?** Determine the net force for the general case where N is any integer, odd or even, greater than one. Proceed as follows: Assume that the total force is not zero. Then it must point in some particular direction. Let every child move one position clockwise. Give a reason that the total force must then have a direction turned clockwise by $360°/N$. Argue that the total force must nevertheless be the same as before. Explain that the contradiction proves that the magnitude of the force is zero. This problem illustrates a widely useful technique of proving a result "by symmetry"—by using a bit of the mathematics of *group theory*. The particular situation

is actually encountered in physics and chemistry when an array of electric charges (ions) exerts electric forces on an atom at a central position in a molecule or in a crystal.

64. A rectangular parallelepiped has dimensions a, b, and c, as in Figure P3.64. (a) Obtain a vector expression for the face diagonal vector $\mathbf{R}_1$. What is the magnitude of this vector? (b) Obtain a vector expression for the body diagonal vector $\mathbf{R}_2$. Note that $\mathbf{R}_1$, $c\hat{\mathbf{k}}$, and $\mathbf{R}_2$ make a right triangle and prove that the magnitude of $\mathbf{R}_2$ is $\sqrt{a^2 + b^2 + c^2}$.

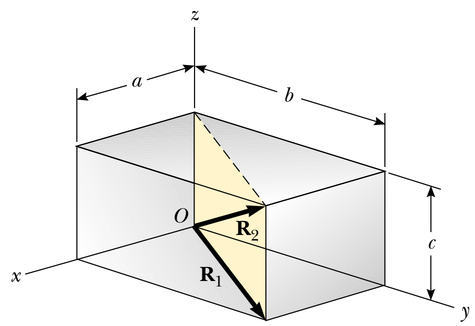


Figure P3.64

65. Vectors $\mathbf{A}$ and $\mathbf{B}$ have equal magnitudes of 5.00. If the sum of $\mathbf{A}$ and $\mathbf{B}$ is the vector $6.00\hat{\mathbf{j}}$, determine the angle between $\mathbf{A}$ and $\mathbf{B}$.

66. In Figure P3.66 a spider is resting after starting to spin its web. The gravitational force on the spider is 0.150 newton down. The spider is supported by different tension forces in the two strands above it, so that the resultant vector force on the spider is zero. The two strands are perpendicular to each other, so we have chosen the x and y directions to be along them. The tension T_x is 0.127 newton. Find (a) the tension T_y, (b) the angle the x axis makes with the horizontal, and (c) the angle the y axis makes with the horizontal.

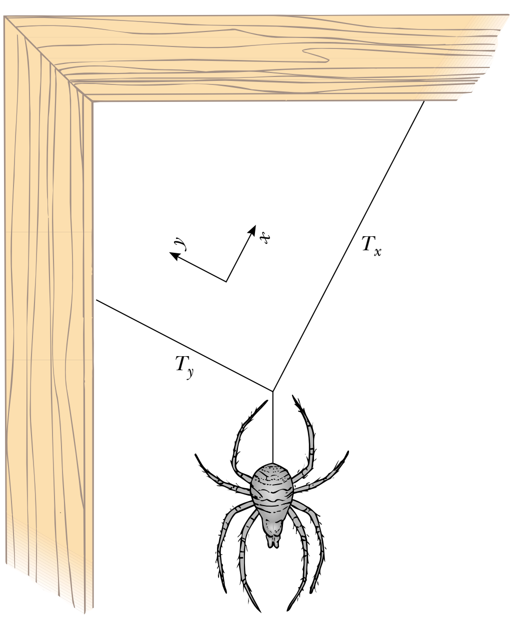


Figure P3.66

67. A point P is described by the coordinates (x, y) with respect to the normal Cartesian coordinate system shown in Fig. P3.67. Show that (x', y'), the coordinates of this point in the rotated coordinate system, are related to (x, y) and the rotation angle α by the expressions

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

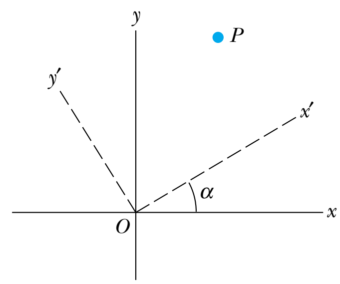


Figure P3.67

Answers to Quick Quizzes

3.1 Scalars: (a), (d), (e). None of these quantities has a direction. Vectors: (b), (c). For these quantities, the direction is necessary to specify the quantity completely.

3.2 (c). The resultant has its maximum magnitude $A + B = 12 + 8 = 20$ units when vector $\mathbf{A}$ is oriented in the same direction as vector $\mathbf{B}$. The resultant vector has its minimum magnitude $A - B = 12 - 8 = 4$ units when vector $\mathbf{A}$ is oriented in the direction opposite vector $\mathbf{B}$.

3.3 (a). The magnitudes will add numerically only if the vectors are in the same direction.

3.4 (b) and (c). In order to add to zero, the vectors must point in opposite directions and have the same magnitude.

3.5 (b). From the Pythagorean theorem, the magnitude of a vector is always larger than the absolute value of each component, unless there is only one nonzero component, in which case the magnitude of the vector is equal to the absolute value of that component.

3.6 (b). From the Pythagorean theorem, we see that the magnitude of a vector is nonzero if at least one component is nonzero.

3.7 (d). Each set of components, for example, the two x components A_x and B_x, must add to zero, so the components must be of opposite sign.

3.8 (c). The magnitude of $\mathbf{C}$ is 5 units, the same as the z component. Answer (b) is not correct because the magnitude of any vector is always a positive number while the y component of $\mathbf{B}$ is negative.